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A Generalization of Hartog's Theorem Relating to the Subalgebras of Continuous Functions on the n-Dimensional Torus

RICHARD TAYLOR

Department of Mathematics, Temple University, Philadelphia, Pennsylvania 19122

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The polydisk algebra, A, generated by the n-coordinate functions, $\pi_1,..., \pi_n$, on D^n (D denotes the unit disk) identifies with a closed subalgebra of $C(T^n)$ (T denotes the unit circle) and it is natural to study those closed subalgebras of $C(T^n)$ containing A. Unlike the case for $n = 1$ (here, A is maximal in $C(T)$ by Wermers maximality theorem), there are many algebras lying properly in between A and $C(T^n)$ for $n > 1$. Although a reasonable characterization of these algebras appears to be exceedingly difficult, the characterization of those whose maximal ideal spaces imbed "nicely" inside D^n is accomplished by Theorems I and I′. Theorem I also generalizes Hartog's classical extension theorem [3; Theorem 5, Chap. I, Sect. C, p. 20], giving rise to a large class of "extension" subsets of D^n, of which its topological boundary is but one example. Other examples relating to a certain class of subalgebras of $C(T^n)$ containing A occur in Section iii.

1. PRELIMINARIES

If B is a commutative Banach algebra, as usual, we let M_B denote the maximum ideal space of B, that is, the set of multiplicative linear functionals on B. For $f_1,..., f_n \in B$, we define their joint spectrum, $\sigma_B(f_1,..., f_n)$, to be $\{(\hat{f}_1(h),..., \hat{f}_n(h)): h \in M_B\}$, where $\hat{f}$ denotes the Gelfand transform of f, $\hat{f}: M_B \to \mathbb{C}$, defined by $\hat{f}(h) = h(f)$. Clearly, $M_B \subset B^*$ (the dual space of B) from which it inherits the weak-* topology. The Silov boundary of B, written ∂B, denotes the (unique) least closed subset of M_B on which each $\hat{f}$ achieves its supremum.

We define, for any compact subset K of $\mathbb{C}^n$, $\mathfrak{A}(K)$ to be the set of functions on K having analytic continuation to a neighborhood of K. Let

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$A(K)$ denote its uniform closure in $C(K)$, the space of continuous functions on K.

A crucial tool to be employed in this paper is the functional calculus: If $f_1,\ldots, f_n \in B$, B a Banach-algebra, and $F \in \mathfrak{A}\ (\sigma_B(f_1,\ldots, f_n))$, then there is an $f \in B$ such that $\hat{f} = F \circ (\hat{f}_1,\ldots, \hat{f}_n)$ [5; Theorem 4.1, Chap. III.4, p. 77]. If B is a function algebra, then f is unique.

Now we sight an application of the functional calculus to be employed in Section ii. If $\pi_1,\ldots, \pi_n$ are the coordinate functions on D^n, $P(D^n)$ is the least closed subalgebra of $C(D^n)$ containing them, it is well known that $D^n = M_{P(D^n)}$ and $T^n = \partial P(D^n)$. Thus $P(D^n)$ identifies with a subalgebra of $C(T^n)$. If

$$P(D^n) \subset B \subset C(T^n), \tag{1}$$

where B is a closed subalgebra of $C(T^n)$, and

$$\pi\colon M_B \to D^n \tag{2}$$

is the canonical projection: $\pi(h) = h_{|P(D^n)}$, then $\sigma_B(\pi_1,\ldots, \pi_n) = \pi(M_B)$. Thus the functional calculus gives .

LEMMA I. *For B and π as above, $A(\pi M_B) \subset B$. In particular, we have $A(D^n) = P(D^n)$, obtained by letting $B = P(D^n)$.*

2. THE MAIN RESULT

For $\varepsilon = (\varepsilon_1,\ldots, \varepsilon_n)$ ($\varepsilon_j = \pm 1$ for all j) define the subalgebra B_ϵ, of $C(T^n)$, to be the one generated by $A(D^n)$ and $\{\pi_j^{\varepsilon j}\colon j = 1,\ldots, n\}$. Such a subalgebra, of which there are exactly 2^n, will be called "distinguished." A standard fact about the spectrum of a function tells us that

$$\begin{aligned} \sigma_{B_\epsilon}(\pi_j) &= D, \qquad \varepsilon_j = 1 \\ &= T, \qquad \varepsilon_j = -1 \end{aligned}$$

By definition B_ϵ is an algebra of rational functions on the solid torus $\mathsf{X}_{j=1}^n$ $\sigma_{B_\epsilon}(\pi_j)$. It is easy to see that, therefore, $f \cdot \pi = \hat{f}$ for each $f \in B_\epsilon$. Thus π imbeds M_{B_ϵ} onto this solid torus. Lemma I thus gives

$$B_\epsilon = A\left(\mathop{\mathsf{X}}_{j=1}^{n} \sigma_{B_\epsilon}(\pi_j)\right). \tag{*}$$

Suppose $T^n \subset K \subset D^n$, K compact, and $\partial A(K) = T^n$. Then $A(K)$ identifies with a subalgebra of $C(T^n)$. Now we state our first theorem, whose proof is postponed until Section IV.

THEOREM I. *Let K be as above and define* ε *by*

$$\begin{aligned} \varepsilon_j &= \ \ 1, \qquad \pi_j(K) \cap \operatorname{Int} D \neq \varnothing \\ &= -1, \qquad \pi_j(K) \cap \operatorname{Int} D = \varnothing. \end{aligned}$$

Then $A(K) = B_\epsilon$.

It follows from (*), for each $f \in A(K)$, there is a unique $\tilde{f} \in A(\mathsf{X}_{j=1}^n \sigma_{B_\epsilon}(\pi_j))$ such that $\tilde{f}_{|K} = f$. In particular, if $K \cap \operatorname{Int} D^n \neq \varnothing$ then $A(K) = A(D^n)$ so that each $f \in A(K)$ has a unique analytic continuation to all of D^n. This is in contrast to the case $K = \operatorname{bdry} D^n$, from which we conclude the same, thus obtaining the special case of Hartog's theorem cited at the onset.

Now fix B as in (1), π as in (2). We say that M_B imbeds conformally inside D^n if for each $f \in B$, there is an $h \in A(\pi M_B)$ such that the diagram

$$\begin{tikzcd} & \mathbb{C} & \\ M_B \arrow[ur, "\hat{f}"] \arrow[rr, "\pi"'] & & \pi M_B \arrow[ul, "h"'] \end{tikzcd}$$

commutes. Lemma I and the fact that $\partial \hat{B} = T^n$ imply $\partial A(\pi M_B) = T^n$. Thus Theorem I applies to give

THEOREM I′. $A(\pi M_B) = B_\epsilon$, *where* $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ *is defined as in Theorem* I *with* $K = \pi M_B$. *In particular,* M_B *imbeds conformally inside* D^n *if, and only if,* B *is distinguished.*

Proof. The first statement is Theorem I. Now, if M_B imbeds conformally inside D^n, then $B_\epsilon \circ \pi = A(\pi M_B) \circ \pi = \hat{B}$, forcing $B = B_\epsilon$. The converse follows directly from arguments preceding Theorem I.

3. A CLASS OF SUBALGEBRAS OF $\mathbb{C}(T^n)$

If $\Sigma \subset Z^n$ satisfies $I, J \in \Sigma \rightarrow I + J \in \Sigma$ (here, $J = (J_1, \ldots, J_n)$ and addition is pontwise), then Σ is an additive semigroup. Thus $\{\pi_1^{J_1} \cdots \pi_n^{J_n} : J \in \Sigma\}$ is a multiplicative semigroup of characters on T^n. Let $A(\Sigma)$ denote the smallest closed subalgebra of $C(T^n)$ containing this semigroup of characters. If $h \in M_{A(\Sigma)}$, then $\tilde{h}: \Sigma \rightarrow D$, defined by $\tilde{h}(J) = h(\pi_1^{J_1} \cdots \pi_n^{J_n})$, is a homomorphism from Σ into D (D is viewed here as a multiplicative subsemigroup of the complex plane). In fact, we have

$$M_{A(\Sigma)} = \operatorname{Hom} \Sigma,$$

the semigroup of all homomorphisms $h: \Sigma \to D$. The most general version of this result, in the context of locally compact abelian groups, can be found in [1; Theorems 4.0, 4.1, pp. 382–383]. For a concrete proof, see [4; Sect. 3(i), p. 37, Theorem II.2.2, p. 34].

Now assume $(Z^+)^n \subset \Sigma \subset Z^n$. $A(\Sigma)$ is distinguished whenever Σ is: that is, whenever Σ is generated by the union of the two sets

$$\{(\delta_1,..., \delta_n): \delta_i = 0 \text{ for all but one value of } i\}$$

and

$$\{(\varepsilon_1,..., \varepsilon_n): \varepsilon_i = \pm 1\}.$$

Otherwise $A(\Sigma)$ is not distinguished. [For instance, let $\Sigma \subset Z^2$ be generated by (0, 1), (1, 0), and (−1, 1)—then $A(\Sigma)$, generated by π_1, π_2, and π_2/π_1, is not of the distinguished sort.] One easily sees that $\pi M_{A(\Sigma)} = \overline{S_\Sigma}$, where $s_\Sigma = \{\zeta \in D^n: \zeta_j \neq 0\ (j = 1,..., n),\ |\zeta_1|^{J_1} \cdots |\zeta_n|^{J_n} \leqslant 1 \text{ for each } J \in \Sigma\}$. Now Theorem I′, interpreted as an extension theorem, specializes to say that if Σ is not distinguished then each $f \in A(S_\Sigma)$ has a unique analytic continuation to all of D^n.

4. Proof of Theorem I

For $K \subset \mathbb{C}^n$, K compact, the automorphisms of $A(K)$ are the bijections, $\varphi: K \to K$, for which $f \circ \varphi^{\pm 1} \in A(K)$ for each $f \in A(K)$.

The automorphisms of D, denoted Aut D, are given by

$$\varphi(z) = \alpha \frac{z - \zeta}{z\bar{\zeta} - 1},$$

where $|\alpha| = 1$, $|\zeta| < 1$ (see [2, Proposition 6.2, p. 184]). Thus if $\varphi_j(z) = \alpha_j((z_j - \zeta_j)/(z_j\bar{\zeta}_{j-1}))$, $|\alpha_j| = 1$, $|\zeta_j| < 1$, then

$$\varphi(z) = (\varphi_1(z),..., \varphi_n(z)) \tag{$*$}$$

defines an automorphism of D^n.

Note that $T_x(z) = (z_1 e^{ix},..., z_n e^{ix})$ is defined for all complex x and z, and $\varphi^{\pm 1}$ both have trivial analytic continuation to the domain $N_r = \{z \in \mathbb{C}^n: |z_j| < 1 + r, j = 1,..., n\}$ for some sufficiently small $r > 0$. There exist r_1, r_2,

$0 < r_1 < r_2 < r$, and $\varepsilon > 0$ such that $\varphi(N_{r_1}) \subset N_{r_2}$ and $T_x(N_{r_2}) \subset N_r$ for each $x \in V_\epsilon$, where $V_\epsilon = \{x \in \mathbb{C}: |\operatorname{Im} x| < \varepsilon\}$. It follows that the function

$$R \times T^n \to T^n,$$
$$(x, z) \to \varphi^{-1}(T_x(\varphi(z)))$$

extends trivially to an analytic fuction

$$V_\epsilon \times N_{r_1} \xrightarrow{\Phi} \mathbb{C}^n.$$

If f is defined and analytic on a neighborhood U_f of T^n, the function

$$\frac{f(\Phi(x, z)) - f(\Phi(w, z))}{x - w}$$

is defined and analytic on the nonempty open set

$$\Phi^{-1}(U_f) \cap \{(x, z) \in \mathbb{C}^{n+1}: x \neq w\}$$

containing $(R - \{w\}) \times T^n$. We use Riemann's extension theorem [3, Theorem 3, Chap. I, Sec. C, p. 19] to conclude that it has unique analytic continuation to a function

$$Q_f: \Phi^{-1}(U_f) \to \mathbb{C}.$$

It follows that

$$\text{for } w \in V_\epsilon, f \in \mathfrak{A}(T^n), \quad \lim_{\substack{x \to w \\ x \in \mathbb{C} - \{w\}}} \frac{f(\Phi(x, z)) - f(\Phi(w, z))}{x - w} = \lim_{x \to w} Q_f(x, z) \quad \text{(A)}$$

is approached *uniformly* for $z \in T^n$.

If we choose φ in (*) so that $\varphi(\tau) = 0$ for a fixed $\tau \in \operatorname{Int} D^n$, then

$$\phi(x) = \varphi^{-1} \circ T_x \circ \varphi \qquad (x \text{ real}) \qquad (**)$$

defines a one-parameter subgroup of Aut D^n, namely, the "translations about τ." Henceforth, we denote this group by Aut_τ. Note that for $x \in R$, $z \in T^n$, $\Phi(x, z) = \phi(x)(z)$. Thus (A) says that the homomorphism (**) from R onto Aut_τ is "analytic" in the sense that for each $f \in \mathfrak{A}(T^n)$ the function

$$x \to f \circ \phi(x) \qquad (x \text{ real}) \qquad (***)$$

is a Banach space valued analytic function from R into $C(T^n)$, where we view $C(T^n)$ as a Banach space in the uniform norm.

Now we are ready to prove Theorem I.

Proof of Theorem I. It is easy to show that the distinguished subalgebras of $C(T^n)$ are precisely those left invariant under composition with elements of Aut_τ for each $\tau \in \mathrm{Int}\, D^n$.[1] With K as in Theorem I, we first show that $A(K)$, identified with a subalgebra of $C(T^n)$, is so invariant, and thus distinguished. To this end, we choose $\tau \in \mathrm{Int}\, D^n$, ϕ as in (**), and $f \in \mathfrak{A}(K)$.

There is an $\varepsilon > 0$ such that $f \circ \phi(x) \in \mathfrak{A}(K)$ for $|x| < \varepsilon$. Since (***) is analytic from R into $C(T^n)$, it follows that $f \circ \phi(x) \in \overline{\mathfrak{A}(K)} = A(K)$ for all $x \in R$.[2] Since $A(K)$ is generated by such functions f, we have $A(K) \circ \phi(x) \subset A(K)$ for all $x \in R$. But $\{\phi(x): x \in R\} = \mathrm{Aut}_\tau$, and we have shown $A(K)$ is left invariant under composition with each of these automorphisms for each $\tau \in \mathrm{Int}\, D^n$. Thus $A(K) = B_\epsilon$ for some n-tuple $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$.

It remains to show that

$$\begin{aligned} \varepsilon_j &= 1, \qquad \pi_j(K) \cap \mathrm{Int}\, D \neq \varnothing \\ &= -1, \qquad \pi_j(K) \cap \mathrm{Int}\, D = \varnothing. \end{aligned}$$

By the definition of B_ϵ, we have $\varepsilon_j = -1$ if and only if $\pi_j^{-1} \in B_\epsilon$ (otherwise. $\varepsilon_j = 1$). Because we showed $B_\epsilon = A(K)$ and $\partial A(K) = T^n$, this holds if and only if $|z_j| = |\pi_j(z)| = 1$ for each $z \in K$, i.e., $\pi_j(K) \cap \mathrm{Int}\, D = \varnothing$. The proof is complete.

Appendix 1

We show here that if $A(D^n) \subset B \subset C(T^n)$, B a closed subalgebra, then B is distinguished (i.e., of the form B_ϵ, defined at the beginning of Section ii) if and only if

$$B \circ \mathrm{Aut}_\tau \subset B \tag{$*$}$$

for each $\tau \in \mathrm{Int}\, D^n$. Although this can be verified directly by synthisizing "characters" in B by the standard averaging process over the groups Aut_τ, we take a function theoretic approach.

If B is distinguished, (*) is trivial to verify. Thus, suppose (*) holds for an algebra B. Let $K = \pi(M_B) \subset D^n$. Then $\varphi(K) \subset K$ for each $\varphi \in \mathrm{Aut}_\tau$, $\tau \in \mathrm{Int}\, D^n$. Thus K is forced to be of the form $\times_{j=1}^n K_{\epsilon_j}$, where

$$\begin{aligned} K_{\epsilon_j} &= T, \qquad \varepsilon_j = -1 \\ &= D, \qquad \varepsilon_j = 1, \qquad j = 1, \ldots, n. \end{aligned}$$

[1] See Appendix 1.

[2] This is the principle of analytic continuation, applied to the Banach space valued analytic function $x \to f \circ \phi(x)$ $(x \in R)$. See Appendix 2.

In particular, we have $K = M_{B_\epsilon}$, yielding

$$M_{B_\epsilon} = \pi M_B. \tag{A.1}$$

But $\pi_j^{-1} \in B$ if and only if π_j does not vanish on K, i.e., $0 \notin \pi_j(K) = K_{\epsilon_j}$, which in turn holds exactly when $\varepsilon_j = -1$. Thus we have

$$B_\epsilon \subset B. \tag{A.2}$$

Now we appeal to the *relative maximality* of B_ϵ inside $C(T^n)$: if (A.1) and (A.2) hold, then $B = B_\epsilon$. This is a corollary of Wermer's classical *maximality* theorem which says that $A(D)$ is maximal inside $C(T)$. These two results are of independent interest, and can be found in [6; Theorem 2.2.2, p. 31] and [5; Theorem 5.1, Chap. II, Sect. 5, p. 38], respectively. [Note: in [6] only the relative maximality of $A(D^n)$ inside $C(T^n)$ is treated. The modifications necessary to treat the general case are strictly academic.]

Appendix 2

Here we generalize the uniqueness principle of analytic continuation of a complex valued analytic function.

THEOREM. *Fix a normed linear space B, an open connected subset $\Omega \subset \mathbb{C}$, and assume $f: \Omega \to B$ is weakly analytic, that is, $\phi \circ f: \Omega \to \mathbb{C}$ is analytic for each $\phi \in B^*$. [This assumption easily follows from the assumption that f is strongly analytic, as was verified for the function* (***) *in Section* (iv).] *If $A \subset B$, A a closed subspace, and $f(E) \subset A$, where E has a limit point in Ω, then $f(\Omega) \subset A$.*

Proof. Let $\langle S \rangle$ denote the closed linear span of $S \subset B$. We need only show that $\langle f(E) \rangle = \langle f(\Omega) \rangle$. Fix $\phi \in B^*$. If $\phi(f(z)) = 0$ for each $z \in E$, then $\phi \circ f: \Omega \to \mathbb{C}$ is a complex valued analytic function vanishing on E, a set with a limit point in the connected domain Ω, from which we conclude from the ordinary uniqueness principle that $\phi(f(z)) = 0$ for all $z \in \Omega$. Since $\phi \in B^*$ was chosen arbitrarily, the Hahn–Banach theorem applies to complete the proof.

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